METHOD OF ALTERNATING DIRECTIONS FOR THE SOLUTION OF PARABOLIC EQUATIONS WITH A CONVECTIVE TERM

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For the solution of the differential equation describing the process of heat propagation in an oil stratum we propose a monotonic difference scheme of alternating directions which converges with a rate of $O(h^2 + \tau^2)$.

1. Statement of the Problem

The mathematical problem may be stated in the following way.

In the rectangular cylinder $\bar{D} = (\Delta + \Gamma) \times [0 < t \le T]$, where $\Delta = \{0 < x_\alpha < l_\alpha,\ \alpha = 1,\ 2\}$, we are required to find a solution of the equation

$$\gamma(x) \frac{\partial u}{\partial t} = Lu + f(x,\ t), \quad x \in \Delta, \quad x = (x_1,\ x_2), \tag{1}$$

which satisfies the boundary condition

$$u|_\Gamma = g(x,\ t) \tag{2}$$

and the initial condition

$$u(x,\ 0) = u_0(x). \tag{3}$$

Here

$$Lu = \sum_{\alpha=1}^{2} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha(x) \frac{\partial u}{\partial x_\alpha}\right) + v_\alpha(x) \frac{\partial u}{\partial x_\alpha},$$

$\gamma(x) \ge c_0 > 0$, $\lambda_\alpha(x) \ge c > 0$, c_0 and c are constants.

2. Difference Scheme

We cover the domain $\Delta + \Gamma$ with a uniform rectangular mesh $\bar{\omega}_k = \{x_{i\alpha} = i_\alpha h_\alpha,\ i_\alpha = 0,\ 1, \ldots,\ N_\alpha,\ N_\alpha = l_\alpha/h_\alpha,\ \alpha = 1,\ 2\}$. Let ω_h denote the set of interior nodes of the mesh $\bar{\omega}_h$. We introduce a mesh for the time coordinate t in a similar way:

$$\bar{\omega}_\tau = \{t^k = 2k\tau, \quad k = 0,\ 1, \ldots,\ N_3, \quad N_3 = T/2\tau\}.$$

On the mesh $\bar{\omega}_h \times \bar{\omega}_\tau$ we approximate the differential problem (1)-(3) by means of a difference problem. For $x \in \bar{\omega}_h \times \bar{\omega}_\tau$ we may write the equations

$$\rho \frac{y^{2k+1} - y^{2k}}{\tau} = \Lambda_1 y^{2k+1} + \Lambda_2 y^{2k} + \varphi^{2k+1}, \tag{4}$$

$$\rho \frac{y^{2k+2} - y^{2k+1}}{\tau} = \Lambda_1 y^{2k+1} + \Lambda_2 y^{2k+2} + \varphi^{2k+1}. \tag{5}$$

To Eqs. (4) and (5) we must append the initial condition

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$$y(x,\ 0) = u_0(x), \qquad x \in \bar{\omega}_h \tag{6}$$

and the difference boundary conditions

$$y^{2k+1} = \frac{1}{2}\,(g^{2k} + g^{2k+2}) - \frac{\tau}{2}\Lambda_2\,(g^{2k+2} - g^{2k}) \tag{7}$$

for $x_1 = 0 (i_1 = 0)$, $x_1 = l_1 (i_1 = N_1)$,

$$y^{2k+2} = g^{2k+2} \tag{8}$$

for $x_2 = 0 (i_2 = 0)$, $x_2 = l_2 (i_2 = N_2)$.

We obtain the condition (7) by subtracting Eq. (5) from Eq. (4) and noting that $y^{2k} = g^{2k}$, $y^{2k+2} = g^{2k+2}$ for $x_1 = 0$ and $x_1 = l_1$.

In Eqs. (4)-(8) we have used the notation

$$x = x_i = (x_{i_1} = i_1 h_1,\ x_{i_2} = i_2 h_2), \quad y = y_{i_1 i_2},$$
$$\rho = \rho_{i_1 i_2}, \quad \varphi^{2k+1} = f(x,\ t^{2k} + \tau), \tag{9}$$
$$\Lambda_\alpha y = R_\alpha (d_\alpha y_{\bar{x}_\alpha})_{x_\alpha} + \left(\frac{\nu_\alpha + |\nu_\alpha|}{2\lambda_\alpha}\right) d^{(+1_\alpha)}\, y_{x_\alpha} + \left(\frac{\nu_\alpha - |\nu_\alpha|}{2\lambda_\alpha}\right) d^{(-1_\alpha)}\, y_{\bar{x}_\alpha},$$

where

$$R_{\alpha;i_\alpha} = \frac{2\lambda_{\alpha;i_\alpha}}{2\lambda_{\alpha;i_\alpha} + h_\alpha |\nu_{\alpha;i_\alpha}|}; \quad \rho_{i_1 i_2} = \gamma(x_{1;i_1},\ x_{2;i_2});$$

$$(d_\alpha y_{\bar{x}_\alpha})_{x_\alpha} = \frac{\lambda_{\alpha;i_\alpha+1/2}\, y_{i_\alpha+1} - (\lambda_{\alpha;i_\alpha+1/2} + \lambda_{\alpha;i_\alpha-1/2})\, y_{i_\alpha} + \lambda_{\alpha;i_\alpha-1/2}\, y_{i_\alpha-1}}{h_\alpha^2}$$

is a second-order divided difference;

$$y_{x_\alpha} = \frac{y_{i_\alpha+1} - y_{i_\alpha}}{h_\alpha}; \quad y_{\bar{x}_\alpha} = \frac{y_{i_\alpha} - y_{i_\alpha-1}}{h_\alpha};$$

$$d^{(+1_\alpha)} = d_{i_\alpha+1} = \lambda_{\alpha;i_\alpha+1/2}; \quad d^{(-1_\alpha)} = d_{i_\alpha} = \lambda_{\alpha;i_\alpha-1/2}, \quad \alpha = 1,\ 2.$$

In this notation we have omitted, for simplicity, one of the subscripts, which for a given direction α is fixed.

A difference operator of the form (9) was used for the first time in [2] to formulate a monotonic, locally uniform scheme converging with a rate of $O(h^2 + \tau)$.

The alternating direction scheme gives a much more accurate solution for one and the same time step than does the locally uniform scheme, although the computational stability of the locally uniform scheme is somewhat better. Because of this the alternating direction scheme can be used with a fairly crude time step when making calculations on an electronic digital computer.

To implement the scheme (4)-(8) we can use the method of paths [1]. Indeed, we may rewrite Eqs. (4) and (5) in the form

$$a_{1;i_1 i_2}\, y^{2k+1}_{i_1-1,i_2} - c_{1;i_1 i_2}\, y^{2k+1}_{i_1 i_2} + b_{1;i_1 i_2}\, y^{2k+1}_{i_1+1,i_2} = -\frac{\rho_{i_1 i_2}}{\tau}\, y^{2k}_{i_1 i_2} - \Lambda_2 y^{2k}_{i_1 i_2} - \varphi^{2k+1}_{i_1 i_2}, \tag{10}$$

$$a_{2;i_1 i_2}\, y^{2k+2}_{i_1,i_2-1} - c_{2;i_1 i_2}\, y^{2k+2}_{i_1 i_2} + b_{2;i_1 i_2}\, y^{2k+2}_{i_1,i_2+1} = -\frac{\rho_{i_1 i_2}}{\tau}\, y^{2k+1}_{i_1 i_2} - \Lambda_1 y^{2k+1}_{i_1 i_2} - \varphi^{2k+1}_{i_1 i_2}, \tag{11}$$

$$i_1 = 1,\ 2,\ \ldots,\ N_1 - 1, \qquad i_2 = 1,\ 2,\ \ldots,\ N_2 - 1,$$

where

$$a_{\alpha;i_\alpha i_\beta} = \left[\left(\frac{R_\alpha}{h_\alpha^2} - \frac{\nu_\alpha - |\nu_\alpha|}{2h_\alpha \lambda_\alpha}\right)_{i_\alpha} \lambda_{\alpha;i_\alpha - \frac{1}{2}}\right]_{i_\beta},$$

$$b_{\alpha;i_\alpha i_\beta} = \left[\left(\frac{R_\alpha}{h_\alpha^2} + \frac{\nu_\alpha + |\nu_\alpha|}{2h_\alpha \lambda_\alpha}\right)_{i_\alpha} \lambda_{\alpha;i_\alpha + \frac{1}{2}}\right]_{i_\beta},$$

$$c_\alpha = a_\alpha + b_\alpha + \frac{\rho}{\tau}, \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta.$$

Since $a_\alpha > 0$, $b_\alpha > 0$, $c_\alpha > a_\alpha + b_\alpha$, then, according to [1], the method of paths is stable, for arbitrary steps h_1 and h_2, for Eq. (10) with the conditions (7) and for Eq. (11) with the conditions (8).

The problem of selection of a method for approximating stationary one-dimensional equations of the form (1) with an error of $O(h^2)$ was treated in detail in [4]. The schemes presented there, involving "central differences" and "noncentral differences," are monotonic, i.e., the maximum principle is valid for these schemes only for very small values of the convective speed. Thus, for example, for a central difference scheme in which the derivative $\nu_\alpha \partial u/\partial x_\alpha$ is approximated by a central difference derivative, the maximum principle is valid and it is solvable by the method of paths only when the conditions $h_\alpha < 2\lambda_\alpha / |\nu_\alpha|$, $\alpha = 1, 2$ are satisfied.

In real problems the coefficient ν_α may prove to be so large that this condition is almost never satisfied.

3. Error of Approximation in the Alternating Direction Method

Let $u = u(x, t)$ be a solution of the differential problem (1)-(3), and let $y^{2k+2} = y^{2k+2}_{i_1 i_2}$ be the solution of the difference problem (4)-(8). As usual, we shall assume that the problem (1)-(3) has the unique solution $u = u(x, t)$ and that all the continuous derivatives of the functions u, λ_α, ν_α required in the course of the development exist.

We consider the differences $z^{2k} = y^{2k} - u^{2k}$, $z^{2k+1} = y^{2k+1} - u^{2k+1}$.

Using Taylor series expansions, we may readily show that

$$u^{2k+1} = \frac{u^{2k+2} - u^{2k}}{2} + O(\tau^2). \tag{12}$$

We substitute $y^{2k} = z^{2k} + u^{2k}$ and $y^{2k+1} = z^{2k+1} - (u^{2k} + u^{2k+2})/2 + O(\tau^2)$ into Eq. (4), and also $y^{2k+2} = z^{2k+2} + u^{2k+2}$ and the expression (12) into Eq. (5); then to ascertain the error incurred in the scheme (4)-(8), we have the problem

$$\rho \frac{z^{2k+1} - z^{2k}}{\tau} = \Lambda_1 z^{2k+1} + \Lambda_2 z^{2k} + \psi_1^{2k+2}, \tag{13}$$

$$\rho \frac{z^{2k+2} - z^{2k+1}}{\tau} = \Lambda_1 z^{2k+1} + \Lambda_2 z^{2k+2} + \psi_2^{2k+2}, \tag{14}$$

$$z^{2k+1} = -\frac{\tau}{2} \Lambda_2 (g^{2k+2} - g^{2k}) \tag{15}$$

for $x_1 = 0 (i_1 = 0)$, $x_1 = l_1 (i_1 = N_3)$,

$$z^{2k+2} = 0 \tag{16}$$

for $x_2 = 0 (i_2 = 0)$, $x_2 = l_2 (i_2 = N_2)$,

$$z(x, 0) = 0. \tag{17}$$

Here ψ_1 and ψ_2 are the approximation errors of schemes (4) and (5), respectively:

$$\psi_1^{2k+2} = \Lambda_1 \frac{u^{2k} + u^{2k+2}}{2} + \Lambda_2 u^{2k} - \rho \frac{u^{2k+2} - u^{2k}}{2\tau} + \varphi^{2k} + O(\tau^2),$$

$$\psi_2^{2k+2} = \Lambda_1 \frac{u^{2k} + u^{2k+2}}{2} + \Lambda_2 u^{2k+2} - \rho \frac{u^{2k+2} - u^{2k}}{2\tau} + \varphi^{2k} + O(\tau^2).$$

The sum $\psi = \psi_1 + \psi_2$ is then the approximation error of the whole scheme (4)-(8).

We now calculate $\Lambda_\alpha u$. To do this we write it in the form

$$\Lambda_\alpha u = (R_\alpha - 1)(d_\alpha u_{\bar{x}_\alpha})_{x_\alpha} + \left(\frac{\nu_\alpha + |\nu_\alpha|}{2\lambda_\alpha}\right) d^{(+1_\alpha)} y_{x_\alpha} + \left(\frac{\nu_\alpha - |\nu_\alpha|}{2\lambda_\alpha}\right) d^{(-1_\alpha)} y_{\bar{x}_\alpha}. \tag{18}$$

Using Taylor series expansion, we obtain

$$(d_\alpha u_{\bar{x}_\alpha})_{x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + O(h_\alpha^2), \tag{19}$$

$$d^{(+1_\alpha)} y_{x_\alpha} = \lambda_\alpha \frac{\partial u}{\partial x_\alpha} + \frac{h_\alpha}{2} \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + O(h_\alpha^2), \tag{20}$$

$$d^{(-1_\alpha)} y_{\bar{x}_\alpha} = \lambda_\alpha \frac{\partial u}{\partial x_\alpha} - \frac{h_\alpha}{2} \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + O(h_\alpha^2). \tag{21}$$

Substituting Eqs.(19)-(21) into Eq.(18), we obtain

$$\Lambda_\alpha u = (R_\alpha - 1) \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + \nu_\alpha \frac{\partial u}{\partial x_\alpha} + \frac{|\nu_\alpha| h_\alpha}{2\lambda_\alpha} \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + O(h_\alpha^2)$$

$$= L_\alpha u - \frac{|\nu_\alpha| h_\alpha}{2\lambda_\alpha + |\nu_\alpha| h_\alpha} \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + \frac{|\nu_\alpha| h_\alpha}{2\lambda_\alpha} \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + O(h_\alpha^2)$$

$$= Lu + \left(\frac{|\nu_\alpha| h_\alpha}{2\lambda_\alpha}\right)^2 \frac{\partial}{\partial x_\alpha}\left(\lambda_\alpha \frac{\partial u}{\partial x_\alpha}\right) + O(h_\alpha^2) = L_\alpha u + O(h_\alpha^2). \tag{22}$$

Using Eq.(22) and the fact that $(u^{2k+2} - u^{2k})/2\tau = \partial u^{2k+1}/\partial t + O(\tau^2)$, we may write

$$\psi_1^{2k+2} = L_1 \frac{u^{2k} + u^{2k+2}}{2} + L_2 u^{2k} - \rho \frac{\partial u^{2k+1}}{\partial t} + f^{2k+1} + O(h^2 + \tau^2)$$

$$= \frac{1}{2}\left(Lu^{2k} - \rho \frac{\partial u^{2k}}{\partial t} + f^{2k}\right) + \frac{1}{2}\left(Lu^{2k+2} - \rho \frac{\partial u^{2k+2}}{\partial t} + f^{2k+2}\right) + \frac{1}{2}(L_2 u^{2k} - L_2 u^{2k+2}) + O(h^2 + \tau^2). \tag{23}$$

Since the expressions in the first two parentheses are zero, we have

$$\psi_1^{2k+2} = \frac{1}{2}(L_2 u^{2k} - L_2 u^{2k+2}) + O(h^2 + \tau^2). \tag{24}$$

Here $h = \max(h_1, h_2)$.

Similarly, we may calculate

$$\psi_2^{2k+2} = \frac{1}{2}(L_2 u^{2k+2} - L_2 u^{2k}) + O(h^2 + \tau^2). \tag{25}$$

Adding Eqs.(24) and (25), we have

$$\psi^{2k+2} = \psi_1^{2k+2} + \psi_2^{2k+2} = O(h^2 + \tau^2);$$

we have thus proved the following theorem.

THEOREM 1. The alternating directions scheme, defined by Eqs.(4)-(8), possesses a total approximation of second order with respect to the space and time steps of the mesh.

4. Stability and Convergence of the Alternating Directions Method

In the proof given in [3] of the convergence of the alternating directions scheme the permutability of the operators Λ_1 and Λ_2 is essential. In our case permutability of the operators Λ_1 and Λ_2 is invalid. We have succeeded in demonstrating convergence of the difference scheme through the use of the maximum principle.

We write the error z in the form $z^{2k} = v^{2k}$, $z^{2k+1} = v^{2k+1} + w^{2k+1}$, $z^{2k+2} = v^{2k+2}$, where $w^{2k+1} = -\tau\Lambda_2(u^{2k+2} - u^{2k})/2$ is given for $0 \le x_1 \le l_1$, $0 < x_2 < l_2$. Now from the relations (13)-(17) we obtain the following problem for v:

$$\rho \frac{v^{2k+1} - v^{2k}}{\tau} = \Lambda_1 v^{2k+1} + \Lambda_2 v^{2k} + \varkappa^{2k+2}, \tag{26}$$

$$\rho \frac{v^{2k+2} - v^{2k+1}}{\tau} = \Lambda_1 v^{2k+1} + \Lambda_2 v^{2k+2} + \varkappa^{2k+2}, \tag{27}$$

$$v^{2k+1}=0$$

for $x_1 = 0(i_1 = 0)$, $x_1 = l_1(i_1 = N)$,

$$v^{2k}=0, \quad v^{2k+2}=0 \tag{29}$$

for $x_2 = 0(i_2 = 0)$, $x_2 = l_2(i_2 = N_2)$,

$$v(x,\ 0)=0. \tag{30}$$

Here

$$\varkappa_1^{2k+2}=\psi_1^{2k+2}-\frac{\rho}{\tau}w^{2k+1}+\Lambda_1 w^{2k+1}=\Lambda_1 w^{2k+1}+O(h^2+\tau^2),$$

$$\varkappa_2^{2k+2}=\psi_2^{2k+2}-\frac{\rho}{\tau}w^{2k+1}+\Lambda_1 w^{2k+1}=\Lambda_1 w^{2k+1}+O(h^2+\tau^2).$$

We then have the following theorem.

THEOREM 2. The solution of the difference problem (4)-(8) converges uniformly to the solution of the differential problem (1)-(3) at a rate equal to the order of the approximation, i.e., at a rate of $O(h^2 + \tau^2)$.

We rewrite Eqs. (26)-(30) in the form

$$\begin{gathered}a_{1;i_1i_2}v_{i_1-1,i_2}^{2k+1}-c_{1;i_1i_2}v_{i_1i_2}^{2k+1}+b_{1;i_1i_2}v_{i_1+1,i_2}^{2k+1}=-F_{1;i_1i_2}^{2k+2},\\ i_1=1,\ 2,\ \ldots,\ N_1-1,\\ v_{i_1i_2}^{2k+1}=0,\ \ i_1=0,\ N_2, \qquad i_2=1,\ 2,\ \ldots,\ N_2-1,\end{gathered} \tag{31}$$

$$\begin{gathered}a_{2;i_1i_2}v_{i_1,i_2-1}^{2k+2}-c_{2;i_1i_2}v_{i_1i_2}^{2k+2}+b_{2;i_1i_2}v_{i_1,i_2+1}^{2k+2}=-F_{2;i_1i_2}^{2k+2}\\ i_2=1,\ 2,\ \ldots,\ N_2-1,\\ v_{i_1i_2}=0,\ \ i_2=0,\ N_2, \qquad i_1=1,\ 2,\ \ldots,\ N_1-1,\end{gathered} \tag{32}$$

where

$$F_1^{2k+2}=\varkappa_1^{2k+2}+\Lambda_2 v^{2k}+\frac{\rho}{\tau}v^{2k},$$

$$F_2^{2k+2}=\varkappa_2^{2k+2}+\Lambda_1 v^{2k+1}+\frac{\rho}{\tau}v^{2k+1}.$$

For difference equations of the form (31), (32) with homogeneous boundary conditions, a maximum principle [1] is available providing that $a_\alpha > 0$, $b_\alpha > 0$, $c_\alpha \geq a_\alpha + b_\alpha$, $\alpha = 1,\ 2$. Therefore, in accord with [1], we have the following inequalities for the solutions of Eqs. (31) and (32):

$$\max_{\overline{\omega}_h}|v^{2k+1}|\leqslant\max_{\omega_h}\left|\frac{F_1^{2k+2}}{\delta}\right|, \tag{33}$$

$$\max_{\overline{\omega}_h}|v^{2k+2}|\leqslant\max_{\omega_h}\left|\frac{F^{2k+2}}{\delta}\right|,\quad \delta=\frac{\rho}{\tau}. \tag{34}$$

We introduce a norm for the vectors in the following way:

$$\|\eta(x)\|=\max_{x\in\omega_h}|\eta(x)|.$$

Then the inequalities (33) and (34) may be rewritten in the form

$$\|v^{2k+1}\|\leqslant\frac{\tau}{c_0}\|\varkappa_1^{2k+2}\|+\left\|\left(E+\frac{\tau}{\rho}\Lambda_2\right)v^{2k}\right\|, \tag{35}$$

$$\|v^{2k+2}\|\leqslant\frac{\tau}{c_0}\|\varkappa_2^{2k+2}\|+\left\|\left(E+\frac{\tau}{\rho}\Lambda_1\right)v^{2k+1}\right\|. \tag{36}$$

Since Λ_1 and Λ_2 are negative operators, then

$$\left\|\left(E+\frac{\tau}{\rho}\Lambda_2\right)v^{2k}\right\|\leqslant\|v^{2k}\|,$$

TABLE 1. Exact and Numerical Solutions for $x_1 = 1.9$ and $x_2 = 1.9$

t	Exact solutions	Numerical solutions: from the central difference scheme	Numerical solutions: from the monotonic scheme
0,01	7,2201	7,2266113	7,231168
0,02	7,2204	5,394775	7,239625
0,03	7,2209	94,65258	7,246469
0,04	7,2216	—3527,73	7,252359
0,05	7,2225	142911,2	7,257691
0,06	7,2236	—5777671,0	7,262702

$$\left\|\left(E + \frac{\tau}{\rho}\Lambda_2\right) v^{2k+1}\right\| \leqslant \|v^{2k+1}\|$$

and therefore

$$\|v^{2k+1}\| \leqslant \frac{\tau}{c_0}\|\varkappa_1^{2k+2}\| + \|v^{2k}\|, \tag{37}$$

$$\|v^{2k+2}\| \leqslant \frac{\tau}{c_0}\|\varkappa_2^{2k+2}\| + \|v^{2k+1}\|. \tag{38}$$

Substituting inequality (37) into inequality (38) and carrying out the recurrent substitution $\|v^{2k}\|$, $\|v^{2k-2}\|$, ..., we obtain

$$\|v^{2k+2}\| \leqslant \sum_{k'=0}^{k} \frac{\tau}{c_0}\left(\|\varkappa_1^{2k'+2}\| + \|\varkappa_2^{2k'+2}\|\right) + \|v^0\|. \tag{39}$$

We have thus demonstrated the uniform stability of the scheme defined by Eqs. (4)-(8). From inequality (39) there also follows the convergence of the solution of the differential problem (1)-(3) to the solution of the difference problem (4)-(8) at a rate equal to the order of the approximation, since

$$\|\Lambda_1 w^{2k+1}\| = O(h^2 + \tau^2).$$

The validity of the latter equality is easily verified since w^{2k+1} is defined even for $x_1 = 0$, $x_1 = l_1$.

5. Numerical Calculations

To check out the quality of the difference scheme considered here, the known analytic solution of the problem (1)-(3) in the square ($1 \leq x_1, x_2 \leq 2$) with

$$\gamma(x_1, x_2) = x_1 + x_2; \quad \lambda_1(x_1, x_2) = x_1x_2; \quad \lambda_2(x_1, x_2) = x_1 + x_2;$$

$$v_1(x_1) = -\exp(5x_1); \quad v_2(x_1, x_2) = -\exp[5(x_1 + x_2)];$$

$$f(x_1, x_2, t) = 2x_1[t - 2x_2 - 1 + \exp(5x_1)] + 2x_2\{t - 2 + \exp[5(x_1 + x_2)]\};$$

$$g(x_1, x_2, t) = \begin{cases} x_2^2 + t^2 + 1 & \text{for } x_1 = 1, \\ x_2^2 + t^2 + 4 & \text{for } x_1 = 2, \\ x_1^2 + t^2 + 1 & \text{for } x_2 = 1, \\ x_1^2 + t^2 + 4 & \text{for } x_2 = 2, \end{cases} \qquad u_0(x_1, x_2) = x_1^2 + x_2^2,$$

was compared with the numerical solutions obtained from the central difference scheme and from the monotonic scheme.

Results of the calculations at the point [1.9, 1.9] are shown in Table 1, the mesh steps for the variables x_1 and x_2 being taken equal to 0.1.

This example shows that the monotonic scheme of alternate directions can be successfully used to calculate processes described by differential equations of the form (1), whereas a strong instability develops at large convective speeds when the central difference scheme and other nonmonotonic schemes are employed.

The scheme presented here may be used for solving quasilinear equations of the form (1) and it may be readily generalized to the case of plane-parallel geometry.

NOTATION

- D is the region in which the solution is sought;
- Γ is the boundary of the region;
- x_1, x_2 are space coordinates;
- t is the time;
- u is the function sought;
- L is the differential operator;
- $\bar{\omega}_h$ is the mesh in space variables;

h_1, h_2	are steps in the mesh;
$h = \max(h_1, h_2)$;	
N_1, N_2	are the number of steps of the mesh $\bar{\omega}_h$ along the axes x_1, x_2, respectively;
$\bar{\omega}_\tau$	is the mesh for the time coordinate;
τ	is the time step;
N_3	is the number of steps along the time axis;
Λ_1, Λ_2	are difference operators;
y	is the mesh analog of function u;
z	is the error;
ψ	is the error of approximating the differential problem by a difference scheme.

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